

## Stress and strain controlled cyclic loading

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### ABSTRACT

*In this paper introducing a nonlinear constitutive integral equation with singular kernels the authors have obtained the stress (strain) response by cyclic imposed strains (stresses). The Mullins and Payne effects are taken into account using a damping function related with the initial damage and nonlinearity using the Ogden equation by large strains. Comparisons between theory and experimental data in both cases (strain and stress controlled tests) are made.*

*Keywords: singular kernels, stress (strain) responses, Mullins, Payne effects, damping function, Ogden equation.*

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### INTRODUCTION

Resinous materials and rubbers are elastoviscous solids. They are very deformable and possess non-linear viscous behaviors. Their creep and stress relaxation is non-linear according to the applied stresses (strains) [1 - 3]. The last non-linearity can be observed excluding the time from the creep curves (the so-called isochrones). Thus, rubbers by large strains require identification and description of nonlinear elastoviscosity. Here we examine polyisoprene rubber with included fillers (cinders) [4].

The Payne effect is an amplitude dependent softening phenomenon. It is characterized by a decrease in the storage modulus with increasing strain amplitude [5].

It is well known that the Boltzmann hereditary theory using integral equations of Volterra [2, 3] can describe the creep and stress relaxation of different viscoelastic solids. In the case of large strains one need to take into account both the physical and geometrical nonlinearities. In this study an analytical approach to obtain the stress or strain response by cycling loading of rubbers at large strains is proposed.

### GENERAL FRAMEWORK

Assuming similarity to the isochrones of the stress relaxation curves, let introduce in the case of nonlinear elastoviscous behavior the following integral equation in order to describe the mechanical behavior of such a rubbers [1 - 3]

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$$\sigma(t) = \varphi(\varepsilon(t)) - \int_0^t R(t-\tau) \varphi(\varepsilon(\tau)) d\tau \quad (1)$$

$\sigma(t)$  is the stress as a function of the time  $t$ ,  $\varepsilon(t)$  is the imposed strain,  $R(t-\tau)$  is the relaxation kernel which can be found from stress relaxation tests and  $\varphi(\varepsilon(t))$  is the instantaneous stress-strain curve. To well describe this curve one can apply the Ogden relation, which in the uniaxial (traction) test can be expressed as [6]

$$\varphi(\varepsilon) = \sum_{i=1}^3 \mu_i (\lambda(\varepsilon)^{\kappa_i-1} - \lambda(\varepsilon)^{-\frac{\kappa_i-1}{2}}) \quad (2)$$

Here  $\mu_1, \mu_2, \mu_3, \kappa_1, \kappa_2, \kappa_3$  are parameters obtained from instantaneous stress-strain test and the stretch  $\lambda$  is related with the engineering strain as follows  $\lambda(\varepsilon) = 1 + \varepsilon$ . Equation (1) with (2) represent the stress response in the case of imposed strain law.

The solution of equation (1) can be represented as follows [7, 8]

$$\varphi(\varepsilon(t)) = \sigma(t) + \int_0^t K(t-\tau) \sigma(\tau) d\tau \quad (3)$$

To obtain the strain curve (nonlinear creep) one should use the inverse function [2,8]

$$\psi(\varepsilon(t)) = \varphi^{-1}(\varepsilon(t)) \quad (4)$$

Equation (4) with (3) represent the strain response in the case of imposed stress law.

In this paper the following stress relaxation kernel is used

$$R(t) = \sum_{i=1}^3 A_i \frac{e^{-\beta_i t}}{t^{\alpha_i}} \quad (5)$$

In this case the solution has the form (3) with the following creep kernel [8]

$$K(t) = \frac{e^{-\beta_i t}}{t} \sum_{n=1}^{\infty} A_i \Gamma(\alpha_i)^n t^{\alpha_i n} / \Gamma(\alpha_i n) \quad (6)$$

$\Gamma(\alpha)$  is the gamma function.  $A_i, \beta_i, \alpha_i$  are ker-

nel parameters. In equations (2, 3) these kernel parameters can be identified from stress relaxation tests.

To take into account the Mullins effect (stress softening after the first cycle) it is introduced in the nonlinear integral equation the so called normalized damping function related as mentioned in [9] with the initial damage by large strains

$$g_o(\varepsilon) = 1 - C \frac{1}{1 + \varepsilon} \quad (7)$$

Here  $C$  is damping parameter which should take into account the damage on the end of the first cycle and being related with some physical parameters (Kachanov's damage [10,11]). As mentioned in [9] this damping function should remain constant when unloading. Thus, the nonlinear constitutive integral equation (the stress response by imposed strains) becomes

$$\sigma(t) = \chi(\varepsilon(t)) - \int_0^t R(t-\tau) \chi(\varepsilon(\tau)) d\tau \quad (8)$$

where

$$\chi(\varepsilon(t)) = \varphi(\varepsilon_{imp}(t)) g(\varepsilon_{imp}(t))$$

and

$$g(\varepsilon_{imp}(t)) = g_o(\varepsilon_{imp}(t)) H(T/2 - t) + g_o(\varepsilon_{imp}(T/2)) H(t - T/2) \quad (9)$$

In this equation  $H(t)$  is the Heaviside function, which helps to retain the damping function constant -  $g_o(\varepsilon_{imp}(T/2))$  after the first loading from 0 to the half period  $T/2$  (the loading time at the end of the first cycle). In the case of sinusoidal impulse loading the imposed period  $T$  is related with the imposed angular frequency  $\omega$  as  $T = 2\pi / \omega$  (see equation (16)).

In the case of stress controlled test one needs the solution of the nonlinear equation (8). Thus, using equation (6) to the strain response we can write (see equations (3, 4))

$$\varepsilon(t) = \chi^{-1}(\sigma(t))$$

here

$$y(t) = \sigma_{imp}(t) + \int_0^t K(t-\tau) \sigma_{imp}(\tau) d\tau \quad (10)$$

Here  $\varepsilon(t) = \chi^{-1}(y(t))$  is the inverse function of  $\chi(y)$  and  $K(t-\tau)$  is the creep kernel which has the form (6). The inverse function can be approximated with parabolic function of arbitrary degree.

The damping parameter  $C$  was obtained calculating the damage ( $d_{exp} = 0.135$ ) from the stored energies after the first and the second cycle using the method proposed in [10,11]

$$d_{exp} = 1 - \frac{U_2(C)}{U_1(C)} \quad (11)$$

Identifying  $C$  we obtain  $\chi(y)$  and his inversion. Thus, from equations (8,10) we arrive to the stress and the strain responses respectively.

Concerning the Payne effect we need to obtain a relation between the storage module and the imposed strain (stress) amplitudes respectively. To do this, we expand the respective responses in Fourier series and after summation of the first three, respectively two members we arrive to the desired relations. In the case of imposed strains the Fourier expansion looks like [12]

$$\sigma(t) / \varepsilon_o = E''_o + \sum_{n=1}^{\infty} E''_n \cos n\omega t + E'_n \sin n\omega t \quad (12)$$

$$\text{with } E''_o = \frac{1}{T} \int_0^T \sigma(t) dt, \quad E''_n = \frac{2}{T} \int_0^T \sigma(t) \cos n\omega t dt$$

$$\text{and } E'_n = \frac{2}{T} \int_0^T \sigma(t) \sin n\omega t dt.$$

Supposing convergence of the Fourier coefficients can be defined the storage module as

$$E'(\varepsilon_o) = \sum_{n=1}^{\infty} E'_n(\varepsilon_o) \quad (13)$$

In the case of imposed stresses the expansion is

$$\varepsilon(t) / \sigma_o = S''_o + \sum_{n=1}^{\infty} S''_n \cos n\omega t + S'_n \sin n\omega t \quad (14)$$

with

$$S''_o = \frac{1}{T} \int_0^T \varepsilon(t) dt, \quad S''_n = \frac{2}{T} \int_0^T \varepsilon(t) \cos n\omega t dt,$$

$$\text{and } S'_n = \frac{2}{T} \int_0^T \varepsilon(t) \sin n\omega t dt,$$

and the storage compliance is defined as

$$S'(\sigma_o) = \sum_{n=1}^{\infty} S'_n(\sigma_o). \quad (15)$$

The convergence in this cases is guaranteed.

In [12] one has proved that the sum  $\sum_{n=1}^{\infty} |b_n|$  converges and in the case of positive  $b_n$  the sum  $\sum_{n=1}^{\infty} b_n$  converges too. Here  $b_n$  are the Fourier coefficients related with the storage or compliance

module according to definitions (13) and (15) respectively. The fact that an integral hereditary equation can take into account the Payne effect is discussed in [13].

## RESULTS AND DISCUSSION

In this work a polyisoprene rubber [4] produced in the UCTM of Sofia is used. The kernel parameters are respectively:

$$A_1 = 0.031, A_2 = 0.001, A_3 = 0.007, \alpha_1 = 0.19, \\ \alpha_2 = 0.77, \alpha_3 = 0.43, \beta_1 = 0.0062, \beta_2 = 0.08, \beta_3 = 0.039.$$

The Ogden parameters are as follows:

$$\mu_1 = 0.13, \mu_2 = -1.46, \mu_3 = 6.2 \times 10^{-3}, \\ \kappa_1 = 2.35, \kappa_2 = -0.991, \kappa_3 = 5.74$$

and the imposed strain and stress laws are:

$$\varepsilon_{imp}(t) = \varepsilon_o (1 + \sin(\omega t - \pi/2))$$

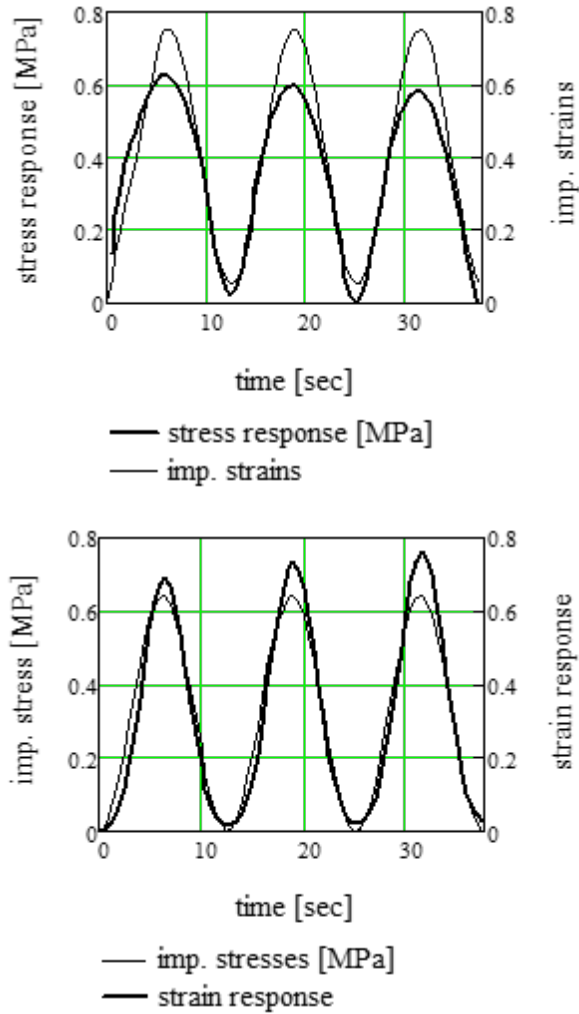


Fig. 1. Imposed impacts and respective responses: up-strain controlled, down - stress controlled.

$$\sigma_{imp}(t) = \sigma_o(1 + \sin(\omega t - \pi/2)) \quad (16)$$

Here  $\varepsilon_o = 0.35$ ,  $\omega = 0.5$  [Hz]. These parameters are chosen in order to respect the large strain amplitude and to avoid the dynamic loading in which case one should take into account the acceleration. The damping parameter is obtained as  $C = 0.3$ . On Fig. 1 it is illustrated the stress (strain) responses in the case of imposed strains (stresses) according to equations (8) and (10), respectively. One can see that in the strain controlled test the stress response diminishes with the time and in the stress controlled case the strain response grows with the time. One can make some analogy

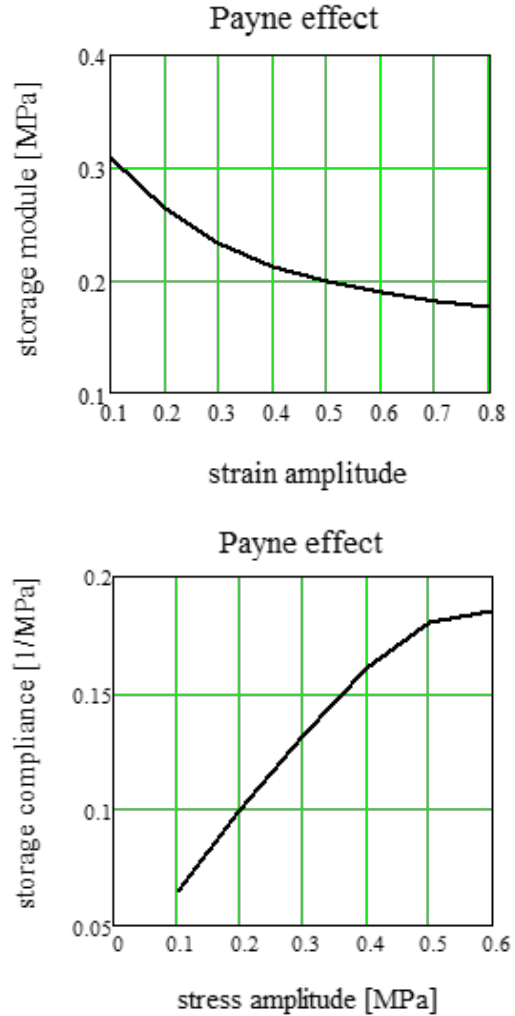


Fig. 2. Payne effect for polyisoprene rubber: up-strain controlled, down - stress controlled.

with the stress relaxation and strain creep cases respectively.

On Fig. 2 is illustrated the Payne softening effect (storage modulus diminution and compliance enhancement) according to the second equations (13, 15).

On the Fig. 3 is illustrated the hysteresis loops for our polyisoprene rubber according to equations (8, 10) and the respective experimental curves obtained with the device described in [14]. One can see the Payne effect too. The respective errors stopping to the third and second terms in the Fourier series do not exceed 6 and 7 %, respectively.

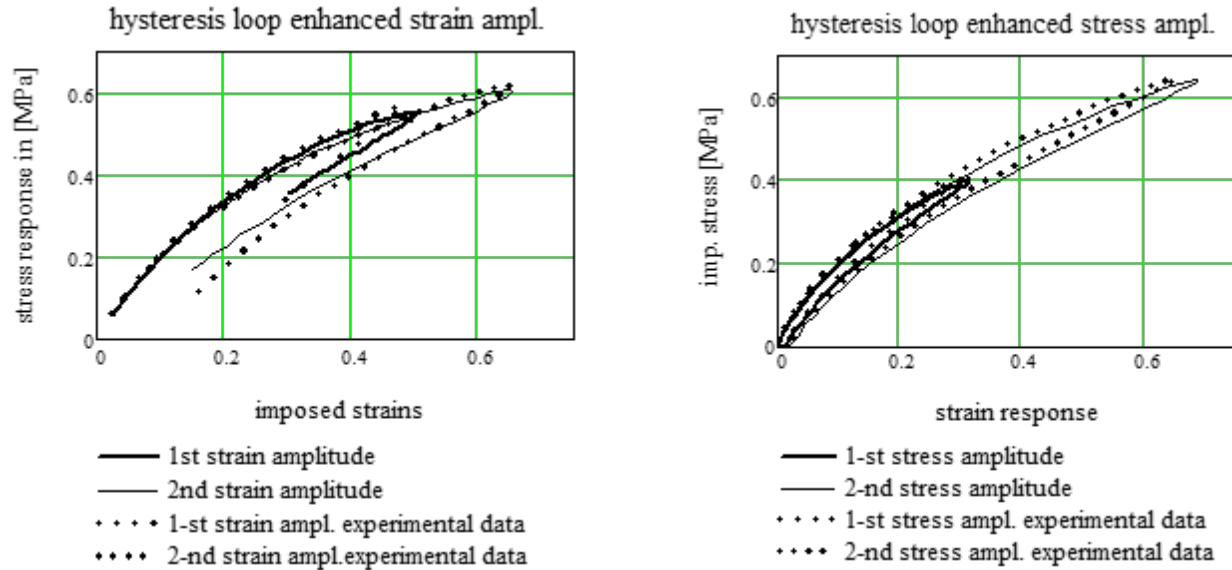


Fig. 3. Hysteresis loops for polyisoprene rubber: left-strain controlled, right-stress controlled.

The Mullins softening effect will be discussed in another work from the same authors.

## CONCLUSIONS

Using nonlinear integral equations with three singular kernels and a damping function whose single parameter can be obtained from independent test, the authors have obtained the stress (strain) responses in the case of imposed strains (stresses) taking into account the Mullins and Payne softening effects. The experimental hysteresis loops and softening curves well agree with the theoretical ones obtained from the stress (strain) responses by imposing sinusoidal pulsations for the strain (stress) laws.

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